A COMBINATORIAL FORMULA FOR THE CHARACTER OF THE DIAGONAL COINVARIANTS

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Abstract. Let R_n be the ring of coinvariants for the diagonal action of the symmetric group S_n. It is known that the character of R_n as a doubly-graded S_n module can be expressed using the Frobenius characteristic map as ∇e_n, where e_n is the n-th elementary symmetric function, and ∇ is an operator from the theory of Macdonald polynomials.

We conjecture a combinatorial formula for ∇e_n and prove that it has many desirable properties which support our conjecture. In particular, we prove that our formula is a symmetric function (which is not obvious) and that it is Schur positive. These results make use of the theory of ribbon tableau generating functions of Lascoux, Leclerc and Thibon. We also show that a variety of earlier conjectures and theorems on ∇e_n are special cases of our conjecture.

Finally, we extend our conjectures on ∇e_n and several of the results supporting them to higher powers $\nabla^m e_n$.

1. Introduction

1.1. Let R_n be the ring of coinvariants for the diagonal action of the symmetric group S_n on $\mathbb{C}^n \oplus \mathbb{C}^n$. In other words,

$$R_n = \mathbb{C}[\mathbf{x}, \mathbf{y}]/I, \tag{1}$$

where $\mathbb{C}[\mathbf{x}, \mathbf{y}] = \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$ is the ring of polynomial functions on $\mathbb{C}^n \oplus \mathbb{C}^n$, the symmetric group acts "diagonally" (*i.e.*, permuting the x and y variables simultaneously), and the ideal $I = ((\mathbf{x}, \mathbf{y}) \cap \mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_n})$ is generated by all S_n-invariant polynomials without constant term. The S_n action respects the double grading

$$R_n = \bigoplus_{r,s} (R_n)_{r,s} \tag{2}$$

given by the x and y degrees.

A formula for the character of R_n as a doubly graded S_n module was conjectured in [5] and proved in [13]. The formula expresses the character in terms of Macdonald polynomials, as follows. Let F denote the Frobenius characteristic: the linear map from S_n characters

Date: October 26, 2003; revised March 2, 2004.

2000 *Mathematics Subject Classification.* Primary: 05E10; Secondary: 05A30, 20C30.

Work supported by an NSA grant (J.H.).

Work supported by NSF Grants DMS-0296203 and DMS-0301072 (M.H.).

Work supported by an NSF Graduate Fellowship (N.L.).

to symmetric functions that sends the irreducible character χ^λ to the Schur function $s_\lambda(z)$. Encoding the graded character of R_n by means of its *Frobenius series*

$$\mathcal{F}_{R_n}(z;q,t) = \sum_{r,s} q^r t^s F\,\mathrm{char}(R_n)_{r,s}, \tag{3}$$

its value is given by the following theorem.

Theorem 1.1.1 ([13]). *Let ∇ be the linear operator defined in terms of the modified Macdonald symmetric functions $\tilde{H}_\mu(z;q,t)$ by*

$$\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu')} \tilde{H}_\mu, \tag{4}$$

where μ is a partition of n, μ' is its conjugate and $n(\mu) = \sum_i (i-1)\mu_i$. Then we have

$$\mathcal{F}_{R_n}(z;q,t) = \nabla e_n(z), \tag{5}$$

where e_n is the nth elementary symmetric function.

The operator ∇ has been the subject of a series of theorems and conjectures of a combinatorial nature [1, 5, 10, 13] (see also [14] for an overview). Specifically, thanks to results of Garsia and Haiman in [5], Theorem 1.1.1 implies that the dimension of R_n is given by

$$\dim_{\mathbb{C}} R_n = (n+1)^{(n-1)}, \tag{6}$$

and that of its subspace R_n^ϵ of S_n-antisymmetric elements by

$$\dim_{\mathbb{C}} R_n^\epsilon = C_n = \frac{1}{n+1}\binom{2n}{n}, \tag{7}$$

the n-th *Catalan number*. These and other related results suggest that we should try to understand the rather mysterious quantity $\nabla e_n(z)$ in more combinatorial terms. Taking a first step in this direction, Garsia and Haglund [3, 4] gave an explicit combinatorial formula for the Hall inner product

$$C_n(q,t) = \langle \nabla e_n, e_n \rangle, \tag{8}$$

which by Theorem 1.1.1 and equation (7) is a q,t-analog of the Catalan number $C_n(1,1) = C_n$. Building on the Garsia-Haglund formula, Haglund and Loehr [9] conjectured a combinatorial formula for the Hilbert series of R_n. By Theorem 1.1.1, this Hilbert series is given by

$$\mathcal{H}_n(q,t) = \langle \nabla e_n, e_1^n \rangle = \sum_{r,s} q^r t^s \dim(R_n)_{r,s}.$$

By [5], it was known that $\mathcal{H}_n(1,t)$ is a generating function enumerating *parking functions* according to a suitably defined weight. The Haglund-Loehr conjecture interprets $\mathcal{H}_n(q,t)$ as a bivariate generating function enumerating parking functions by the usual weight, together with another statistic counting certain kinds of inversions (see §4.5).

In this paper we conjecture a combinatorial formula for the full expansion of $\nabla e_n(z)$ in terms of monomials, generalizing the Garsia-Haglund formula for $C_n(q,t)$, the Haglund-Loehr conjecture for $\mathcal{H}_n(q,t)$, and a conjecture in [2] expressing $\langle \nabla e_n, h_d e_{n-d} \rangle$ in terms of

Schröder paths. We prove that our formula is, as it ought to be, a symmetric function. As will be seen, this property of our formula is not obvious from its definition, but follows from the theory of ribbon tableau generating functions developed by Lascoux, Leclerc and Thibon [19, 20].

By Theorem 1.1.1, $\nabla e_n(z)$ is Schur positive, that is, its coefficients $\langle \nabla e_n(z), s_\lambda \rangle$ on the Schur basis belong to $\mathbb{N}[q,t]$. We prove that our conjectured formula is also, as it ought be, Schur positive. For this, however, we must rely on an interpretation of our formula in terms of Kazhdan-Lusztig polynomials, as in [20]. We are unable as yet to provide a combinatorial interpretation for its Schur function expansion.

Finally, we extend our considerations to higher powers $\nabla^m e_n(z)$, giving corresponding conjectured formulas and examining their properties.

2. Preliminaries

2.1. **q-Series notation.** We use the standard notations:

$$(z;q)_k = (1-z)(1-zq)\cdots(1-zq^{k-1}), \tag{9}$$

$$[k]_q = \frac{1-q^k}{1-q}, \tag{10}$$

$$[k]_q! = (q;q)_k/(1-q)^k = [k]_q\,[k-1]_q\,\cdots\,[1]_q, \tag{11}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = (q^{n-k+1};q)_k/(q;q)_k = \frac{[n]_q!}{[k]_q![n-k]_q!}, \tag{12}$$

$$\begin{bmatrix} n \\ k_1,\ldots,k_r \end{bmatrix}_q = \frac{[n]_q!}{[k_1]_q!\cdots[k_r]_q!}, \quad \text{where } k_1+\cdots+k_r = n. \tag{13}$$

2.2. **Partitions and tableaux.** We represent an integer partition as usual by the sequence

$$\lambda = (\lambda_1, \ldots, \lambda_l)$$

of its parts in decreasing order, and denote its size by

$$|\lambda| = \sum_i \lambda_i.$$

It is understood that $\lambda_i = 0$ for $i > l$. We may also write

$$\lambda = (1^{\alpha_1}, 2^{\alpha_2}, \ldots)$$

to indicate the partition with α_i parts equal to i. The *conjugate partition* λ' is defined by

$$\lambda'_i = \sum_{j \geq i} \alpha_j.$$

The *Young diagram* of λ is the set $\{(i,j) : 0 \leq j < \lambda_{i+1}\} \subseteq \mathbb{N}\times\mathbb{N}$. One pictures elements $(i,j) \in \mathbb{N}\times\mathbb{N}$ as boxes or *cells*, arranged with the i-axis vertical and the j-axis horizontal, so the rows of the diagram are the parts of λ. Abusing notation, we usually write λ both for a partition and its diagram. A *skew* Young diagram λ/μ is the difference of partition

diagrams $\mu \subseteq \lambda$. A skew diagram is a *horizontal strip* (resp. *vertical strip*) if it contains no two cells in the same column (resp. row).

A *semistandard Young tableau* of (skew) shape λ is a function T from the diagram of λ to the ordered alphabet

$$\mathcal{A}_+ = \{1 < 2 < \cdots\}$$

which is weakly increasing on each row of λ and strictly increasing on each column. A semistandard tableau is *standard* if it is a bijection from λ to $\{1, 2, \ldots, n = |\lambda|\}$. More generally, we admit the alphabet

$$\mathcal{A}_\pm = \mathcal{A}_+ \cup \mathcal{A}_- = \{1 < \bar{1} < 2 < \bar{2} < \cdots\}$$

of *positive* letters $1, 2, \ldots$ and *negative* letters $\bar{1}, \bar{2}, \ldots$. A *super* tableau is a function $T\colon \lambda \to \mathcal{A}_\pm$, weakly increasing on each row and column, such that the entries equal to a in T occupy a horizontal strip if a is positive, and a vertical strip if a is negative. Thus a semistandard tableau is just a super tableau with positive entries. We denote

$$\begin{aligned}
\mathrm{SSYT}(\lambda) &= \{\text{semistandard tableaux } T\colon \lambda \to \mathcal{A}_+\}\\
\mathrm{SSYT}_\pm(\lambda) &= \{\text{super tableaux } T\colon \lambda \to \mathcal{A}_\pm\}\\
\mathrm{SSYT}(\lambda, \mu) &= \{\text{semistandard tableaux } T\colon \lambda \to \mathcal{A}_+ \text{ with entries } 1^{\mu_1}, 2^{\mu_2}, \ldots\}\\
\mathrm{SSYT}_\pm(\lambda, \mu, \eta) &= \{\text{super tableaux } T\colon \lambda \to \mathcal{A}_\pm \text{ with entries } 1^{\mu_1}, \bar{1}^{\eta_1}, 2^{\mu_2}, \bar{2}^{\eta_2}, \ldots\}\\
\mathrm{SYT}(\lambda) &= \{\text{standard tableaux } T\colon \lambda \to \{1, \ldots, n = |\lambda|\}\} = \mathrm{SSYT}(\lambda, (1^n)).
\end{aligned}$$

2.3. **Symmetric functions.** We follow the notation of [24], writing e_λ for the elementary symmetric functions, h_λ for the complete homogeneous symmetric functions, m_λ for the monomial symmetric functions, p_λ for the power-sums and s_λ for the Schur functions. We take these in variables $z = z_1, z_2, \ldots$ so as not to confuse them with the variables $\mathbf{x}$, $\mathbf{y}$ in R_n.

We write $\langle -, - \rangle$ for the Hall inner product, defined by either of the identities

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu} = \langle s_\lambda, s_\mu \rangle. \tag{14}$$

We denote by ω the involution defined by any of the identities

$$\omega e_\lambda = h_\lambda; \quad \omega h_\lambda = e_\lambda; \quad \omega s_\lambda = s_{\lambda'}. \tag{15}$$

We use square brackets $f[A]$ to denote the plethystic evaluation of a symmetric function f on a polynomial, rational function or formal series A. This is defined by writing f in terms of power sums and then substituting $p_m[A]$ for p_m, where $p_m[A]$ is the result of substituting $a \mapsto a^m$ for every indeterminate in A. The standard λ-ring identities hold for plethystic evaluation, *e.g.*, $e_n[A + B] = \sum_k e_k[A]e_{n-k}[B]$, and so forth. In particular, setting $Z = z_1 + z_2 + \cdots$, we have $f[Z] = f(z)$ for all f. Using this notation, we may write

$$\omega^W f[Z + W] \tag{16}$$

to denote the result of applying ω to $f[Z + W] = f(z_1, z_2, \ldots, w_1, w_2, \ldots)$, considered as a symmetric function in the w variables with functions of z as coefficients. Equations (14)

and (15) then imply that the coefficient of a monomial $z^\mu w^\eta = z_1^{\mu_1} z_2^{\mu_2} \cdots w_1^{\eta_1} w_2^{\eta_2} \cdots$ in $\omega^W f[Z+W]$ is given by

$$\omega^W f[Z+W] \,|_{z^\mu w^\eta} = \langle f, e_\eta h_\mu \rangle. \tag{17}$$

If T is a semistandard tableau of (skew) shape λ, we set

$$z^T = \prod_{x \in \lambda} z_{T(x)}. \tag{18}$$

Then the familiar combinatorial formula for (skew) Schur functions reads

$$s_\lambda(z) = \sum_{T \in \mathrm{SSYT}(\lambda)} z^T. \tag{19}$$

Throughout what follows we fix

$$Z = z_1 + z_2 + \cdots, \quad W = w_1 + w_2 + \cdots, \tag{20}$$

and make the convention that

$$z_{\bar{a}} \textit{ stands for } w_a, \textit{ for every negative letter } \bar{a} \in \mathcal{A}_\pm.$$

In particular, this means that if $T \in \mathrm{SSYT}_\pm(\lambda, \mu, \eta)$, then $z^T = z^\mu w^\eta$ by definition. The "super" analog of (19) is then

$$\omega^W s_\lambda[Z+W] = \sum_{T \in \mathrm{SSYT}_\pm(\lambda)} z^T, \tag{21}$$

which follows immediately from (17) and the Pieri rule.

2.4. **Quasisymmetric functions.** Let $T(x) = a$, $T(y) = a+1$ be consecutive entries in a standard tableau T, with $x = (i,j)$, $y = (i',j')$. If $j \geq j'$, we say that a is a *descent* of T. The descent set of T is the subset

$$d(T) = \{a : a \text{ is a descent of } T\} \subseteq \{1, \ldots, n-1\}.$$

Given any subset $D \subseteq \{1, \ldots, n-1\}$, Gessel's *quasisymmetric function* is defined by the formula

$$Q_{n,D}(z) = \sum_{\substack{a_1 \leq a_2 \leq \cdots \leq a_n \\ a_i = a_{i+1} \Rightarrow i \notin D}} z_{a_1} z_{a_2} \cdots z_{a_n}. \tag{22}$$

Here the indices a_i belong to the alphabet of positive letters $\mathcal{A}_+$.

Proposition 2.4.1 ([6]). *The (skew) Schur function $s_\lambda(z)$ is given in terms of quasisymmetric functions by*

$$s_\lambda(z) = \sum_{T \in \mathrm{SYT}(\lambda)} Q_{|\lambda|, d(T)}(z).$$

We will need a "super" version of the above proposition. To this end, define "super" quasisymmetric functions

$$\tilde{Q}_{n,D}(z,w) = \sum_{\substack{a_1 \leq a_2 \leq \cdots \leq a_n \\ a_i = a_{i+1} \in \mathcal{A}_+ \Rightarrow i \notin D \\ a_i = a_{i+1} \in \mathcal{A}_- \Rightarrow i \in D}} z_{a_1} z_{a_2} \cdots z_{a_n}. \tag{23}$$

Here the indices a_i range over $\mathcal{A}_\pm$. Note that our convention $z_{\bar{a}} = w_a$ remains in force, so the right-hand side stands for an expression involving both z and w variables. The next proposition generalizes Proposition 2.4.1. We review the well-known proof because later on we shall want to prove similar results for other kinds of tableaux by appealing to the same mode of reasoning.

Proposition 2.4.2. *The (skew) super Schur function $\tilde{s}_\lambda(z,w) = \omega^W s_\lambda[Z+W]$ is given in terms of super quasisymmetric functions by*

$$\tilde{s}_\lambda(z,w) = \sum_{T \in \mathrm{SYT}(\lambda)} \tilde{Q}_{|\lambda|, d(T)}(z,w). \tag{24}$$

In particular, Proposition 2.4.1 follows on setting $w = 0$.

Proof. If ν is a horizontal strip, there is a unique labelling of the cells of ν to form a standard tableau with no descents (namely, label the cells in increasing order by columns). Symmetrically, if ν is a vertical strip, there is a unique standard tableau on ν with descents at every position. From these observations it follows that every super tableau T, say of shape λ, has a unique *standardization* S such that S is standard, $T \circ S^{-1}$ is a weakly increasing function, and if $T \circ S^{-1}(j) = T \circ S^{-1}(j+1) = \cdots = T \circ S^{-1}(k) = a$, then $\{j, \ldots, k-1\} \cap d(S)$ is empty if a is positive, and equal to $\{j, \ldots, k-1\}$ if a is negative.

Conversely, the shape of a standard tableau with no descents can only be a horizontal strip, and symmetrically, the shape of a tableau with descents at every position can only be a vertical strip. It follows that for a given standard tableau S of shape λ, if $T' \colon \{1, \ldots, n\} \to \mathcal{A}_\pm$ is a weakly increasing function that satisfies the conditions above, then $T = T' \circ S$ is a super tableau, and its standardization is equal to S. Since the sum of z^T over all such T is equal to $\tilde{Q}_{|\lambda|, d(S)}(z,w)$, it follows that (24) is just another way of writing (21). □

Corollary 2.4.3. *Let $f(z)$ be any symmetric function homogeneous of degree n, written in terms of quasisymmetric functions as*

$$f(z) = \sum_D c_D Q_{n,D}(z). \tag{25}$$

Then its "superization" $\tilde{f}(z,w) = \omega^W f[Z+W]$ is given by

$$\tilde{f}(z,w) = \sum_D c_D \tilde{Q}_{n,D}(z,w). \tag{26}$$

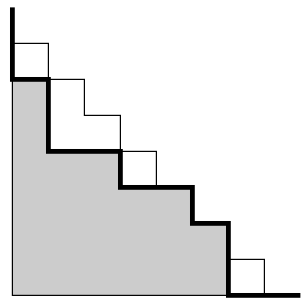

FIGURE 1. A partition $\lambda \subseteq \delta_n$ (shaded) and its Dyck path (heavy line).

Proof. The quasisymmetric functions $Q_{n,D}(z)$ are linearly independent; hence the coefficients c_D are uniquely determined by f, and the right-hand side of (26) depends linearly on f. When f is a Schur function, the result follows from Propositions 2.4.1 and 2.4.2. This implies the result for all f by linearity. $\square$

3. THE MAIN CONJECTURE

3.1. Fix n and let

$$\delta_n = (n-1, n-2, \ldots, 1, 0) \tag{27}$$

be the "staircase" partition. Let $\lambda \subseteq \delta_n$ be a partition whose diagram is contained in the staircase. Note that the outer boundary of λ, together with segments along the i and j-axes, can be identified with a *Dyck path*: a lattice path from $(n, 0)$ to $(0, n)$ by steps of the form $(-1, 0)$ (south) and $(0, 1)$ (east) that never goes above the diagonal line $i + j = n$. Figure 1 illustrates this. The number of Dyck paths, or of partitions $\lambda \subseteq \delta_n$, is the Catalan number C_n.

Let T be a semistandard tableau of skew shape $(\lambda + (1^n))/\lambda$, that is, the vertical strip formed by the cells (i, λ_{i+1}) for $i = 0, 1, \ldots, n-1$. For every cell $x = (i, j) \in \mathbb{N} \times \mathbb{N}$ let $d(x) = i + j$, so $d(x) = k$ means that x is on the k-th diagonal. Given two entries $T(x) = a$ and $T(y) = b$ of T with $a < b$, put $x = (i, j)$, $y = (i', j')$. We say that these two entries form a *d-inversion* if either

(i) $d(y) = d(x)$ and $j > j'$, or
(ii) $d(y) = d(x) + 1$ and $j < j'$.

Set

$$\operatorname{dinv}(T) = \text{number of d-inversions in } T. \tag{28}$$

For example, the tableau T in Figure 2 has $\operatorname{dinv}(T) = 8$, with d-inversions formed by the pairs of entries $(8, 2)$, $(8, 4)$, $(6, 1)$, $(6, 3)$, $(7, 1)$, $(7, 3)$, $(4, 1)$, and $(2, 1)$.

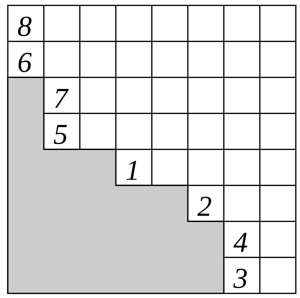

FIGURE 2. A standard tableau of shape $(\lambda + (1_n))/\lambda$.

Definition 3.1.1.

$$D_n(z;q,t) = \sum_{\lambda \subseteq \delta_n} \sum_{T \in \mathrm{SSYT}(\lambda+(1^n)/\lambda)} t^{|\delta_n/\lambda|} q^{\mathrm{dinv}(T)} z^T. \tag{29}$$

Conjecture 3.1.2. *We have the identity*

$$\nabla e_n(z) = D_n(z;q,t). \tag{30}$$

Equivalently, for all μ we have

$$\langle \nabla e_n, h_\mu \rangle = \sum_{\lambda \subseteq \delta_n} \sum_{T \in \mathrm{SSYT}(\lambda+(1^n)/\lambda,\, \mu)} t^{|\delta_n/\lambda|} q^{\mathrm{dinv}(T)}. \tag{31}$$

The bulk of our work in this paper will serve to show that Conjecture 3.1.2 is consistent with previously known or conjectured properties of ∇e_n. The most basic such property is that ∇e_n is a symmetric function.

Theorem 3.1.3. *The quantity $D_n(z;q,t)$ is a symmetric function in z, and it is Schur positive,* i.e., $\langle D_n(z;q,t), s_\mu(z)\rangle \in \mathbb{N}[q,t]$ *for all μ. In fact, each term*

$$D_n^\lambda(z;q) = \sum_{T \in \mathrm{SSYT}(\lambda+(1^n)/\lambda)} q^{\mathrm{dinv}(T)} z^T \tag{32}$$

is individually symmetric and Schur positive.

Note that it is not at all obvious from the definition that $D_n(z;q,t)$ is symmetric. Its symmetry is equivalent to the assertion that the right-hand side of (31) does not depend on the order of the parts of μ. Our proof uses Lascoux, Leclerc and Thibon's theory of spin generating functions for ribbon tableaux. We will give a synopsis of their theory and prove Theorem 3.1.3 in §5. For now, we take the theorem for granted, and explore what can be deduced by more elementary means.

3.2. **Superization.** Our first goal is to show that Conjecture 3.1.2 implies a seemingly stronger formula, giving the superization $\omega^W(\nabla e_n)[Z+W]$. We extend the definition of $\mathrm{dinv}(T)$ to super tableaux T as follows. Let $x, y \in (\lambda + (1^n))/\lambda$ satisfy condition (i) or (ii)

in the definition of d-inversion, above. Then entries $T(x) = a$, $T(y) = b$ with $a, b \in \mathcal{A}_\pm$ form a d-inversion in T if $a < b$ or if $a = b \in \mathcal{A}_-$ is a negative letter.

Theorem 3.2.1. *The superization* $\tilde{D}_n(z, w; q, t) = \omega^W D_n[Z + W; q, t]$ *is given by*

$$\tilde{D}_n(z, w; q, t) = \sum_{\lambda \subseteq \delta_n} \sum_{T \in \mathrm{SSYT}_\pm(\lambda + (1^n)/\lambda)} t^{|\delta_n/\lambda|} q^{\mathrm{dinv}(T)} z^T. \tag{33}$$

Equivalently, Conjecture 3.1.2 implies

$$\langle \nabla e_n, e_\eta h_\mu \rangle = \sum_{\lambda \subseteq \delta_n} \sum_{T \in \mathrm{SSYT}_\pm(\lambda + (1^n)/\lambda,\, \mu,\, \eta)} t^{|\delta_n/\lambda|} q^{\mathrm{dinv}(T)}. \tag{34}$$

Proof. Consider the total ordering of $\mathbb{N} \times \mathbb{N}$ (*reverse diagonal lexicographic order*) defined by

$$x <_d y \quad \text{if } d(x) > d(y), \text{ or } d(x) = d(y) \text{ and } j < j', \text{ where } x = (i, j),\ y = (i', j').$$

On every (skew) shape ν there is a unique standard tableau whose labels are decreasing with respect to $<_d$; and there is a tableau with increasing labels, also unique, if and only if all cells of ν are in distinct rows and columns. Suppose now that ν is contained in $(\lambda + (1^n))/\lambda$. Then ν is a vertical strip *a fortiori*, and it is a horizontal strip if and only if its cells are in distinct rows and columns. Hence on ν there exists a $<_d$-decreasing (resp. $<_d$-increasing) standard tableau if and only if ν is a vertical (resp. horizontal) strip. In either case the tableau in question is unique.

Define a to be a d-descent of a standard tableau $S \in \mathrm{SYT}(\lambda + (1^n)/\lambda)$ if $S(x) = a$, $S(y) = a+1$ with $x >_d y$, and denote by $dd(S)$ the set of d-descents of S. Define the standardization of a super tableau $T \in \mathrm{SSYT}(\lambda + (1^n)/\lambda)$ to be the unique standard tableau S such that $T \circ S^{-1}$ is weakly increasing, and if $T \circ S^{-1}(j) = T \circ S^{-1}(j+1) = \cdots = T \circ S^{-1}(k) = a$, then $\{j, \ldots, k-1\} \cap dd(S)$ is empty if a is positive, and equal to $\{j, \ldots, k-1\}$ if a is negative. The proof of Proposition 2.4.2 again goes through to show that the standardization in this new sense exists, and that the sum $\sum_T z^T$ over all super tableaux T with standardization S is equal to the super quasisymmetric function $\tilde{Q}_{n,dd(S)}(z, w)$.

Note that if cells x, y satisfy condition (i) or (ii) in the definition of d-inversion, then $x >_d y$. In particular, a standard tableau labelled in $<_d$-increasing order has no d-inversions, while one labelled in $<_d$-decreasing order has a d-inversion in every such pair of cells x, y. With this in mind, we see that if T is a super tableau and S its standardization, then $\mathrm{dinv}(S) = \mathrm{dinv}(T)$. This yields the formula

$$\sum_{T \in \mathrm{SSYT}_\pm(\lambda + (1^n)/\lambda)} q^{\mathrm{dinv}(T)} z^T = \sum_{S \in \mathrm{SYT}(\lambda + (1^n)/\lambda)} q^{\mathrm{dinv}(S)} \tilde{Q}_{n,dd(S)}(z, w). \tag{35}$$

Setting $w = 0$, we obtain the quasisymmetric function expansion

$$D_n(z; q, t) = \sum_{\lambda \subseteq \delta_n} \sum_{S \in \mathrm{SYT}(\lambda + (1^n)/\lambda)} t^{|\delta_n/\lambda|} q^{\mathrm{dinv}(S)} Q_{n,dd(S)}(z). \tag{36}$$

By (35), the right-hand side of (33) is the superization of this, and the theorem now follows from Theorem 3.1.3 and Corollary 2.4.3. ∎

3.3. **Shuffle formulation.** Recall that a *parking function* on n cars is a function $f\colon \{1,\dots,n\} \to \{1,\dots,n\}$ satisfying

$$|f^{-1}(\{1,\dots,k\})| \geq k \quad \text{for all } k = 1,\dots,n. \tag{37}$$

For every function $f\colon \{1,\dots,n\} \to \mathbb{N}_{>0}$ there is a unique partition λ with at most n parts and a standard tableau T of shape $(\lambda + (1^n))/\lambda$ such that each entry a of T lies in column $f(a) - 1$. Namely, the parts of λ are the values $f(i) - 1$, and the entries of T in the cells $(i-1, \lambda_i)$ for which $\lambda_i = j - 1$ are the elements of $f^{-1}(\{j\})$. It is easy to see that f is a parking function if and only $\lambda \subseteq \delta_n$.

Let f be a parking function encoded by $\lambda \subseteq \delta_n$ and $T \in \mathrm{SYT}(\lambda + (1^n)/\lambda)$. Reading off the entries of T in $<_d$-increasing order yields a permutation $w(f)$, with descent set

$$D(w(f)^{-1}) = dd(T). \tag{38}$$

For example, for the parking function encoded by the tableau in Figure 2, we have $w(f) = 82467135$ and $D(w(f)^{-1}) = dd(T) = \{1,3,5,7\}$.

Say that a permutation w is a *μ,η-shuffle* if its inverse is the concatenation of alternately increasing and decreasing sequences of lengths $\mu_1, \eta_1, \mu_2, \eta_2, \dots$. Define the *area* $a(f)$ to be $|\delta_n/\lambda| = \binom{n+1}{2} - \sum_i f(i)$; this is equal to the traditional weight of the parking function, as in [5, 9, 10]. Define $\mathrm{dinv}(f) = \mathrm{dinv}(T)$. Then we have the following corollary to Theorem 3.1.3 and the proof of Theorem 3.2.1.

Corollary 3.3.1. *Conjecture 3.1.2 implies that $\langle \nabla e_n, e_\eta h_\mu \rangle$ is the generating function*

$$\sum_f t^{a(f)} q^{\mathrm{dinv}(f)}, \tag{39}$$

summed over parking functions f such that $w(f)$ is a μ,η-shuffle. Even without assuming Conjecture 3.1.2, the above sum is independent of the order of the parts of η and μ.

Remark. The sum in (39) is also independent of the way in which μ and η are interleaved, as can be seen by setting some of the parts in the standard interleaving $\mu_1, \eta_1, \mu_2, \eta_2, \dots$ to zero.

4. Specializations

4.1. **Value at $q = 1$.** By [5, Thms. 2.1, 2.2 and 3.6], we have the formula

$$\nabla e_n(z)|_{q=1} = \sum_{\lambda \subseteq \delta_n} t^{|\delta_n/\lambda|} e_\alpha(z), \tag{40}$$

where $\lambda = (0^{\alpha_0}, 1^{\alpha_1}, 2^{\alpha_2}, \dots)$ with α_0 defined to make $\sum_i \alpha_i = n$. We verify that Conjecture 3.1.2 is consistent with this formula.

Proposition 4.1.1. *We have $D_n(z; 1, t) = \nabla e_n(z)|_{q=1}$.*

Proof. Since the skew Schur function $s_{(\lambda+(1^n))/\lambda}$ is equal to e_α, this is obvious from (40) and Definition 3.1.1. $\square$

Remark. Although $D_n(z;1,t)$ is trivial to evaluate, $D_n(z;q,1)$ is not—see §7, Problem 4.

4.2. **Value at $t=0$.**

Lemma 4.2.1. *We have*

$$\nabla e_n(z)|_{t=0} = (q;q)_n h_n[Z/(1-q)]. \tag{41}$$

Proof. Equation (5) implies that $\nabla e_n(z)|_{t=0}$ is the Frobenius series in one parameter q of the classical coinvariant ring $R_n \cap \mathbb{C}[\mathbf{x}]$. By a result of Stanley [28], this is given by the right-hand side of (41). □

Proposition 4.2.2. *We have*

$$D_n(z;q,0) = \nabla e_n(z)|_{t=0}. \tag{42}$$

Proof. Only the term $\lambda = \delta_n$ contributes when $t=0$. Then every cell $x \in (\lambda + (1^n))/\lambda$ is on the diagonal $d(x) = n$, a tableau $T \in \mathrm{SSYT}(\lambda + (1^n)/\lambda)$ is just a word in the alphabet $\mathcal{A}_+$, and $\mathrm{dinv}(T)$ is its number of inversions in the ordinary sense. Hence

$$D_n(z;q,0) = \sum_\mu \begin{bmatrix} n \\ \mu_1, \dots, \mu_l \end{bmatrix}_q m_\mu(z). \tag{43}$$

By the Cauchy formula,

$$(q;q)_n h_n[Z/(1-q)] = \sum_\mu (q;q)_n h_\mu[1/(1-q)] m_\mu(z), \tag{44}$$

which is equal to the right-hand side of (43). □

4.3. **Value at $q=0$.** We begin by observing that if $\lambda \subseteq \delta_n$ and λ' has distinct parts, then for a parking function f encoded by a tableau $T \in \mathrm{SYT}(\lambda + (1^n)/\lambda)$, the permutation $w(f)$ is obtained simply by reading T from the top row to the bottom. Fix $w_0 \in S_n$ to be the permutation $w_0(i) = n+1-i$, so $w(f)w_0$ is $w(f)$ read backwards.

Lemma 4.3.1. *A standard tableau $T \in \mathrm{SYT}(\lambda + (1^n)/\lambda)$ has $\mathrm{dinv}(T) = 0$ if and only if λ' has distinct parts and the descent set of $w(f)w_0$ is the set of parts of λ', where f is the parking function encoded by T.*

Proof. Suppose λ' does not have distinct parts. Then the associated Dyck path contains two or more consecutive horizontal steps, not on the j-axis, and there is a cell $x \in (\lambda + (1^n))/\lambda$ bordering the vertical step which follows them. There is at least one other cell $y \in (\lambda + (1^n))/\lambda$ with $d(y) = d(x)$ and y to the left of x; fix y to be the rightmost of these. By assumption, y and x are not consecutive on their common diagonal. Hence the cell z directly below y in $\mathbb{N} \times \mathbb{N}$ is also in $(\lambda + (1^n))/\lambda$; otherwise y would not have been the rightmost cell. No matter what the entries of T in the three cells x, y, z are, they form at least one d-inversion.

We have shown that $\mathrm{dinv}(T) = 0$ implies that λ' has distinct parts. Once this is given, it is easy to see that $\mathrm{dinv}(T) = 0$ if and only if, in addition, the descents of $w(f)w_0$ are the parts of λ'. □

Proposition 4.3.2. *We have*

$$D_n(z;0,t) = \nabla e_n(z)|_{q=0}. \tag{45}$$

Proof. Recall that the major index $\operatorname{maj}(w)$ of a permutation or a tableau is defined as the sum of its descents. Every permutation w occurs uniquely as $w(f)$ for a shape λ, tableau T and parking function f satisfying the conditions in Lemma 4.3.1. Moreover, we have $|\delta_n/\lambda| = \operatorname{maj}(w_0 w(f) w_0)$. Lemma 4.3.1 and equations (36) and (38) therefore imply that

$$D_n(z;0,t) = \sum_{w\in S_n} t^{\operatorname{maj}(w_0 w w_0)} Q_{n,D(w^{-1})}(z). \tag{46}$$

Recall that if (P_w, Q_w) is the pair of standard tableaux associated to w by the Schensted correspondence, then $d(Q_w) = D(w)$ and $d(P_w) = D(w^{-1})$. Recall further that $Q_{w_0ww_0} = \operatorname{ev}(Q_w)$, where ev is the *evacuation* operator of Schützenberger [27]. Hence we can rewrite (46) as

$$D_n(z;0,t) = \sum_{(P_w,Q_w)} t^{\operatorname{maj}(\operatorname{ev} Q_w)} Q_{n,d(P_w)}(z). \tag{47}$$

Recall [29, Prop. 7.19.11] that

$$(t;t)_n s_\lambda(1/(1-t)) = \sum_{T\in \mathrm{SYT}(\lambda)} t^{\operatorname{maj}(T)}. \tag{48}$$

Using this and Proposition 2.4.1, we see that the right-hand side of (47) decomposes as

$$\sum_\lambda (t;t)_n s_\lambda(1/(1-t)) s_\lambda(z), \tag{49}$$

which is equal to $\nabla e_n(z)|_{q=0}$ by Lemma 4.2.1 and the Cauchy formula. □

4.4. **The q,t-Catalan formula.** Haglund conjectured [7] and with Garsia proved [3, 4] a formula for the q,t-Catalan polynomial

$$C_n(q,t) = \langle \nabla e_n, e_n\rangle. \tag{50}$$

Haglund and Loehr [9] later showed that the formula of [3, 4] can also be written in the form

$$C_n(q,t) = \sum_{\lambda\subseteq\delta_n} t^{|\delta_n/\lambda|} q^{\operatorname{dinv}(\lambda)}, \tag{51}$$

where $\operatorname{dinv}(\lambda)$ is defined to be $\operatorname{dinv}(T)$ for the super tableau of shape $(\lambda+(1^n))/\lambda$ whose every entry is $\bar{1}$. It is immediate from (34) that this coincides with $\langle D_n(z;q,t), e_n\rangle$.

The formulation of (51) in terms of $\operatorname{dinv}(\lambda)$ was motivated by a conjecture of Haiman, whose original form was slightly different. Namely,

$$C_n(q,t) = \sum_{\lambda\subseteq\delta_n} t^{|\delta_n/\lambda|} q^{b(\lambda)}, \tag{52}$$

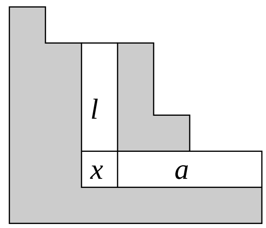

FIGURE 3. The arm and leg of a cell x in a Young diagram.

where $b(\lambda)$ is the number of cells $x \in \lambda$ such that

$$l(x) \leq a(x) \leq l(x) + 1. \tag{53}$$

Here the *arm* $a(x)$ (resp. *leg* $l(x)$) is the number of cells in the hook of x that are in the same row (resp. column) as x, excluding x itself—see Figure 3. To tie (51) and (52) together, let us show that in fact $b(\lambda) = \operatorname{dinv}(\lambda)$.

Lemma 4.4.1. *The number $b(\lambda)$ defined above is equal to the number of pairs of cells $x, y \in (\lambda + (1^n))/\lambda$ satisfying condition (i) or (ii) in the definition of d-inversion in §3.1.*

Proof. Let x be a cell of λ, let u be the cell of $(\lambda + (1^n))/\lambda$ just outside the end of the arm of x, and let t be the cell just outside the end of the leg of x. Travelling along the diagonal $i + j = d(t)$, starting at t and moving in the increasing i direction, let v be the first cell of $(\lambda + (1^n))/\lambda$ encountered (it always exists). Then $a(x) = l(x)$ if and only if u, v satisfy (i) in the definition of d-inversion, while $a(x) = l(x) + 1$ if and only if v, u satisfy (ii). Moreover, every pair of cells satisfying (i) or (ii) arises uniquely in this way. A fully detailed argument in a more general setting will be given in the proof of Lemma 6.3.3, below. ☐

4.5. **The Haglund-Loehr conjecture.** The Hilbert series of R_n is given by $\mathcal{H}_n(q,t) = \langle \nabla e_n, e_1^n \rangle$. Conjecture 3.1.2 implies that this is equal to

$$\langle D_n(z;q,t), e_1^n \rangle = \sum_{\lambda \subseteq \delta_n} \sum_{T \in \mathrm{SYT}(\lambda + (1^n)/\lambda)} t^{|\delta_n/\lambda|} q^{\operatorname{dinv}(T)}. \tag{54}$$

If desired, one may express the same thing as a sum $\sum_f t^{a(f)} q^{\operatorname{dinv}(f)}$ over all parking functions f on n cars. It is none other than the value given for $\mathcal{H}_n(q,t)$ by a conjecture of Haglund and Loehr in [9]. Thus the Haglund-Loehr conjecture is an immediate consequence of Conjecture 3.1.2.

4.6. **Fermionic formula.** The original “fermionic formula” is that of Kerov, Kirillov and Reshetikhin [17, 18] giving the q-Kostka coefficient $K_{\lambda\mu}(q)$ as a sum of products of q-binomial coefficients. By analogy, we use the same terminology for an expansion of a q,t-quantity as a sum of powers of t times products of q-binomial coefficients. Haglund [7] gave a fermionic formula in this sense for $C_n(q,t)$. Here we give a fermionic formula for $\langle D_n(z;q,t), e_\eta h_\mu \rangle$.

Let $\sigma = \sigma_1 \dots \sigma_n \in S_n$ be a permutation with say $k-1$ descents, at positions $r_1 < \cdots < r_{k-1}$, and set $r_0 = 0$, $r_k = n$. Let $A_j = A_j(\sigma) = \{\sigma_l : r_{j-1} + 1 \leq l \leq r_j\}$ be the jth “run”

of σ. Denote by $F(\sigma)$ the set of all parking functions f whose encoding tableau T has the property that A_j is the set of entries of T in cells x on the diagonal $d(x)=n+1-j$. Define

$$H(\sigma;q,t)=\sum_{f\in F(\sigma)} q^{\mathrm{dinv}(f)}t^{a(f)}. \tag{55}$$

With this notation, [9, Thm. 1] can be formulated as follows:

$$H(\sigma;q,t)=t^{\mathrm{comaj}(\sigma)}\prod_{i=2}^{n}[v(\sigma,i)+\chi(i\le r_1)]_q, \tag{56}$$

where $\chi(a\le b)$ equals 1 if $a\le b$ and 0 otherwise, $\mathrm{comaj}(\sigma)=\sum_{i=1}^{k-1}(n-r_i)=\mathrm{maj}(\sigma w_0)$, and $v(\sigma,i)$ is the largest value of p, $1\le p\le i-1$, for which the sequence

$$(\sigma_{i-p}-\sigma_i \bmod n),\ (\sigma_{i-p+1}-\sigma_i \bmod n),\ \ldots,\ (\sigma_{i-1}-\sigma_i \bmod n) \tag{57}$$

is increasing (in other words, $\sigma_{i-p},\ldots,\sigma_i$ is a rotation of an increasing sequence). Hence $\langle D_n(z;q,t),e_1^n\rangle$, *i.e.*, our conjectured value for the Hilbert series $\mathcal{H}_n(q,t)=\langle\nabla e_n,e_1^n\rangle$ of R_n, is the sum of the right-hand side of (56) over all $\sigma\in S_n$.

Let

$$H^{\mu}(\sigma;q,t)=\sum_{f\in F^{\mu}(\sigma)} q^{\mathrm{dinv}(f)}t^{a(f)}, \tag{58}$$

where $F^{\mu}(\sigma)$ is the set of all parking functions $f\in F(\sigma)$ such that $w(f)$ is a $\mu,\emptyset$-shuffle. Note that if $f\in F(\sigma)$, then σ is the permutation obtained by sorting each block of $w(f)$ contributed by the cells on one diagonal $d(x)=n+1-j$ in the tableau T that encodes f (these blocks are the sets A_j). Clearly $H^{\mu}(\sigma;q,t)=0$ if σ is not a $\mu,\emptyset$-shuffle. Otherwise, if σ is a $\mu,\emptyset$-shuffle, define $B_j=\{M_{j-1}+1,\ldots,M_j\}$, where $M_j=\mu_1+\cdots+\mu_j$. In other words, the collection $\{B_j : 1\le j\le l(\mu)\}$ is the partition of $\{1,\ldots,n\}$ into blocks of consecutive integers of sizes μ_j.

Note that if $f\in F^{\mu}(\sigma)$, then the elements of $A_i\cap B_j$ occur in $w(f)$ in increasing order for each i and j. Let $\mathfrak{S}\subseteq S_n$ be the subgroup consisting of permutations τ that map each set $A_i\cap B_j$ into itself. Then, given any $g\in F(\sigma)$, there is a unique $f\in F^{\mu}(\sigma)$ and $\tau\in\mathfrak{S}$ such that $w(g)=\tau\circ w(f)$. Hence,

$$\Big(\prod_{i,j} b_{i,j}!\Big)H^{\mu}(\sigma;1,t)=H(\sigma;1,t), \tag{59}$$

where $b_{i,j}=|A_i\cap B_j|$. Moreover, taking into account the definition of dinv, it is clear that

$$H^{\mu}(\sigma;q,t)=\frac{H(\sigma;q,t)}{\prod_{i,j}[b_{i,j}]_q!}. \tag{60}$$

Now set $V_{i,j}=v(\sigma,k)+\chi(k\le r_1)$, where σ_k is the largest element of $A_i\cap B_j$. Using the fact that the elements of $A_i\cap B_j$ form an increasing sequence of adjacent, consecutive

integers in σ, the definition of $v(\sigma, i)$ implies that

$$\frac{H(\sigma;q,t)}{\prod_{i,j}[b_{i,j}]_q!} = t^{\mathrm{comaj}(\sigma)} \prod_{i,j} \begin{bmatrix} V_{i,j} \\ b_{i,j} \end{bmatrix}_q. \tag{61}$$

Combining this with (60) yields the fermionic formula

$$\langle D_n(z;q,t), h_\mu \rangle = \sum_{\substack{\sigma\in S_n \\ \sigma \text{ is a } \mu,\emptyset\text{-shuffle}}} t^{\mathrm{comaj}(\sigma)} \prod_{i,j} \begin{bmatrix} V_{i,j} \\ b_{i,j} \end{bmatrix}_q. \tag{62}$$

More generally, there is a similar formula for $\langle D_n(z;q,t), e_\eta h_\mu\rangle$. Set $M_j = \mu_1 + \cdots + \mu_j$ as before, and $E_j = \eta_1 + \cdots + \eta_j$. In this setting we redefine $B_j = \{M_{j-1} + E_{j-1} + 1, \ldots, M_j + E_{j-1}\}$ and set $C_j = \{M_j + E_{j-1} + 1, \ldots, M_j + E_j\}$. We define A_j to be the j-th run of σ, just as before. Let $\tilde{\sigma}$ be the permutation obtained by reversing each block $A_i \cap C_j$ in σ.

Put $b_{i,j} = |A_i \cap B_j|$ and $c_{i,j} = |A_i \cap C_j|$. Define $V_{i,j}$ as before, and set $W_{i,j} = v(\sigma, k) + \chi(k \leq r_1)$, where σ_k is the largest element of $A_i \cap C_j$. Then similar reasoning yields the formula

$$\langle D_n(z;q,t), e_\eta h_\mu \rangle = \sum_{\substack{\sigma\in S_n \\ \tilde{\sigma} \text{ is a } \mu,\eta\text{-shuffle}}} t^{\mathrm{comaj}(\sigma)} \Big(\prod_{i,j} \begin{bmatrix} V_{i,j} \\ b_{i,j} \end{bmatrix}_q\Big)\Big(\prod_{i,j} q^{\binom{c_{i,j}}{2}} \begin{bmatrix} W_{i,j} \\ C_{i,j} \end{bmatrix}_q\Big). \tag{63}$$

Remarks. (1) In the above, we do not assume that μ and η are partitions. Thus "the" fermionic formula is really a separate formula for each possible ordering of the parts of μ and η. The symmetry part of Theorem 3.1.3 is equivalent to the statement that all these formulas yield the same result.

(2) The reader may enjoy verifying that the special case $\mu = \emptyset$, $\eta = (n)$ of (63) agrees with the fermionic formula in [7] for the q,t-Catalan polynomial.

4.7. **Schröder paths.** In [2], Egge, Haglund, Killpatrick and Kremer conjectured a combinatorial formula for $\langle \nabla e_n, e_{n-d} h_d\rangle$. In this section we first show how this conjecture is a special case of (34). Then we briefly discuss how some ideas in Haglund's recent proof of their conjecture suggest a refinement of Conjecture 3.1.2.

A *Schröder path* is a lattice path from $(n, 0)$ to $(0, n)$ composed of steps of the form $(-1, 0)$ (south), $(0, 1)$ (east) and $(-1, 1)$ (diagonal) which never goes above the line $i + j = n$.[1] Egge, *et al.* gave two different formulations of their conjecture, one involving a pair of statistics $(\mathrm{area}, \mathrm{bounce})$ on Schröder paths and another involving a pair of statistics $(\mathrm{dinv}, \mathrm{area})$. They showed that the two formulations are equivalent by exhibiting a bijection which sends $(\mathrm{dinv}, \mathrm{area})$ to $(\mathrm{area}, \mathrm{bounce})$.

Given a Schröder path Π, let $\lambda(\Pi)$ be the partition whose associated Dyck path is obtained by replacing each diagonal step of Π by a south step followed by an east step. If Π has d diagonal steps, let $T(\Pi)$ be the super tableau of shape $(\lambda + (1^n))/\lambda$ obtained by placing the number 1 in each square bordered by one of the d new pairs of south, east steps replacing

[1] In [2], partitions are drawn in the fourth quadrant, English style, where we draw them in the first quadrant, French style. Correspondingly, our Schröder paths are mirror images of those in [2].

the former diagonal steps, and setting all other entries to $\bar{1}$. The reader will have no problem checking from the definitions of $\mathrm{dinv}(\Pi)$ and $\mathrm{area}(\Pi)$ given in [2] that $\mathrm{dinv}(\Pi) = \mathrm{dinv}(T(\Pi))$ and $\mathrm{area}(\Pi) = |\delta_n/\lambda(\Pi)|$. Thus the conjecture of Egge, *et al.* is equivalent to the special case of (34) for $e_\eta h_\mu = e_{n-d}h_d$.

More recently, Haglund [8] has proven the conjecture of Egge, *et al.*, and also the $h_\mu = h_{n-d}h_d$ case of (31). An important role in both proofs is played by functions $E_{n,k}$, defined as the coefficients in the Newton interpolation series expansion

$$e_n[Z\frac{1-u}{1-q}] = \sum_{k=1}^{n} \frac{(u;q)_k}{(q;q)_k} E_{n,k}(z). \tag{64}$$

Note that by setting $u = q$ in (64) we see $\sum_{k=1}^{n} E_{n,k} = e_n$. The $E_{n,k}$ were first introduced by Garsia and Haglund in their proof of (51). In [8] it is conjectured that

$$\langle \Delta_{s_\beta} \nabla E_{n,k}, s_\mu \rangle \in \mathbb{N}[q,t] \tag{65}$$

for all μ, β. Here Δ_f is a linear operator defined on the modified Macdonald basis as

$$\Delta_f \tilde{H}_\mu = f[B_\mu]\tilde{H}_\mu, \tag{66}$$

where

$$B_\mu = \sum_{(i,j)\in\mu} t^i q^j. \tag{67}$$

In particular, (65) implies

$$\langle \nabla E_{n,k}, s_\mu \rangle \in \mathbb{N}[q,t]. \tag{68}$$

The conjectured truth of (65) is largely motivated by [13, Cor. 3.5], which implies

$$\langle \Delta_{s_\beta} \nabla e_n, s_\mu \rangle \in \mathbb{N}[q,t]. \tag{69}$$

We now introduce a refinement of Conjecture 3.1.2 which implies (68), namely

Conjecture 4.7.1. *For* $1 \le k \le n$*,*

$$\nabla E_{n,k} = \sum_{\substack{\lambda\subseteq\delta_n \\ |\{i:\lambda_i = n-i\}| = k}} t^{|\delta_n/\lambda|} D_n^\lambda(z;q). \tag{70}$$

5. Proof of Theorem 3.1.3

In this section we will summarize some results of Lascoux, Leclerc and Thibon [19, 20], adding to these a new description of their "spin" statistic on ribbon tableaux, and use all this to prove Theorem 3.1.3.

5.1. **Cores and quotients.** We begin by recalling some standard facts from the combinatorial theory of n-cores and n-quotients, as developed for instance in [15, 30]. An *n-ribbon* is a connected skew shape of size n and depth 1, *i.e.*, containing no 2×2 rectangle. A partition μ is an *n-core* if there is no $\nu \subseteq \mu$ such that μ/ν is an n-ribbon. Every μ contains a unique n-core $\nu = \mathrm{core}_n(\mu)$ such that μ/ν can be tiled by n-ribbons. In other words, if we successively remove as many n-ribbons from μ as possible, the shape ν that remains does not depend on any choices made.

The *content* of a cell $x = (i, j) \in \mathbb{N} \times \mathbb{N}$ is defined as $c(x) = j - i$. Define the content of an n-ribbon to be the maximum of the contents of its cells. If ν is an n-core, there are exactly n shapes μ such that μ/ν is an n-ribbon, and the contents $s_0, s_1, \ldots, s_{n-1}$ of these ribbons are distinct $\pmod n$. We always index them so that $s_i \equiv i \pmod n$. Then the n-cores are in one-to-one correspondence with cosets $(s_0, s_1, \ldots, s_{n-1}) + \mathbb{Z} \cdot (n, n, \ldots, n)$ satisfying $s_i \equiv i \pmod n$ for all i.

Fix an n-core ν with content sequence $(s_0, s_1, \ldots, s_{n-1})$. If $(\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(n-1)})$ is any n-tuple of partitions, we define the *adjusted content* of a cell $x \in \mu^{(i)}$ to be

$$\tilde{c}(x) = nc(x) + s_i. \tag{71}$$

Note that $\tilde{c}(x)$ determines which $\mu^{(i)}$ the cell x belongs to via its congruence class $\pmod n$. Let $\mathcal{P}$ be the set of all partitions, and let $\mathcal{P}_\nu = \{\mu \in \mathcal{P} : \mathrm{core}_n(\mu) = \nu\}$. There is a bijection

$$\mathrm{quot}_n \colon \mathcal{P}_\nu \to \mathcal{P}^n, \tag{72}$$

written $\mathrm{quot}_n(\mu) = (\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(n-1)})$, characterized by the following property: in any tiling of μ/ν by n-ribbons, the multiset of contents of the ribbons is equal to the multiset of adjusted contents $\tilde{c}(x)$, taken over all i and all cells $x \in \mu^{(i)}$. In particular, we have

$$|\mu/\nu| = n\, |\mathrm{quot}_n(\mu)| \underset{\text{def}}{=} n \sum_i |\mu^{(i)}|. \tag{73}$$

For $\lambda, \mu \in \mathcal{P}_\nu$ we also have

$$\lambda \subseteq \mu \quad \Leftrightarrow \quad \lambda^{(i)} \subseteq \mu^{(i)} \text{ for all } i. \tag{74}$$

Therefore, quot_n extends to a bijection $\mathrm{quot}_n(\mu/\lambda) = (\mu^{(0)}/\lambda^{(0)}, \ldots, \mu^{(n-1)}/\lambda^{(n-1)})$ from skew shapes μ/λ with $\lambda, \mu \in \mathcal{P}_\nu$ to n-tuples of skew shapes. To avoid notational ambiguity, we henceforth apply quot_n only to skew shapes, so if μ is a partition with $\mathrm{core}_n(\mu) = \nu$, we would write $(\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(n-1)}) = \mathrm{quot}_n(\mu/\nu)$.

We remark that a given skew shape may have multiple representations $\theta = \mu/\lambda$ with different n-cores $\nu = \mathrm{core}_n(\mu) = \mathrm{core}_n(\lambda)$. However, the resulting n-quotients $\mathrm{quot}_n(\theta)$ differ only by translations of the components $\theta^{(i)}$, which compensate for the change in the contents s_i associated with ν so that the adjusted contents $\tilde{c}(x)$ for $x \in \mathrm{quot}_n(\theta)$ remain the same.

A standard, semistandard or super tableau on an n-tuple of shapes $(\mu^{(0)}, \ldots, \mu^{(n-1)})$ just means a tableau of the specified sort on the disjoint union of the shapes $\mu^{(i)}$. A *standard n-ribbon tableau* on a (skew) shape μ is a tiling of μ by n-ribbons and a function $T \colon \mu \to \{1, 2, \ldots, N = |\mu|/n\}$, weakly increasing on each row and column, which is constant on each

ribbon and induces a bijection from the ribbons to $\{1, 2, \ldots, N\}$. It follows from (74) that quot_n induces a bijection between standard n-ribbon tableaux of shape μ and $\mathrm{SYT}(\mathrm{quot}_n(\mu))$.

Call μ a *horizontal* (resp. *vertical*) *n-ribbon strip* if it can be tiled by n-ribbons and each $\mu^{(i)}$ is a horizontal (resp. vertical) strip. Then there is a unique standard ribbon tableau T of shape μ in which the ribbons are labelled in increasing (resp. decreasing) order of content—namely, $\mathrm{quot}_n(T)$ is the unique standard tableau of shape $\mathrm{quot}_n(\mu)$ in which the cells are labelled in increasing (resp. decreasing) order of adjusted content $\tilde{c}(x)$. The ribbon tiling given by this distinguished tableau is the *official tiling* of the strip μ. In the case of a horizontal ribbon strip μ, the official tiling is characterized by the property that the cell of maximum content in each ribbon is the minimum cell in its column in the shape μ.

A *semistandard n-ribbon tableau* of shape μ is a tiling of μ by n-ribbons and a function $T\colon \mu \to \mathcal{A}_+$, weakly increasing on each row and column and constant on each ribbon, such that for each $a \in \mathcal{A}_+$, $T^{-1}(a)$ is a horizontal n-ribbon strip with the official tiling. Define z^T to be the product of $z_{T(\theta)}$ over the n-ribbons θ in the given tiling of μ, or equivalently, $\prod_{x\in\mu} z_{T(x)} = (z^T)^n$. It is immediate that quot_n induces a weight-preserving bijection between the set $\mathrm{SSRT}^n(\mu)$ of semistandard n-ribbon tableaux of shape μ and $\mathrm{SSYT}(\mathrm{quot}_n(\mu))$.

5.2. **Spin generating functions.** As in [19], the *spin* $s(\theta)$ of an n-ribbon θ is one less than the number of its rows. Given a (semi)standard n-ribbon tableau T, we set $s(T)$ equal to the sum of $s(\theta)$ over all ribbons θ in the tiling underlying T. One proves that $(-1)^{s(T)}$ depends only on the shape of T. Let $\mathrm{smin}(\mu)$ and $\mathrm{smax}(\mu)$ be the minimum and maximum of $s(T)$ over all T of shape μ. The *spin* and *cospin* of a tableau T of shape μ are then defined to be the integers $\mathrm{sp}(T) = \frac{1}{2}(s(T) - \mathrm{smin}(\mu))$ and $\mathrm{csp}(T) = \frac{1}{2}(\mathrm{smax}(\mu) - s(T))$, respectively.

Theorem 5.2.1 ([19])**.** *For every (skew) shape μ, the spin generating function*

$$G_\mu(z; q) = \sum_{T\in \mathrm{SSRT}^n(\mu)} q^{\mathrm{sp}(T)} z^T \tag{75}$$

is a symmetric function.

To apply this theorem in our setting, we need an alternative description of spin. Fix a content sequence $(s_0, s_1, \ldots, s_{n-1})$ with $s_i \equiv i \pmod n$ and an n-tuple of shapes $\boldsymbol{\mu} = (\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(n-1)})$. Let S be a semistandard tableau of shape $\boldsymbol{\mu}$. An *inversion* is a pair of entries $S(x) = a$, $S(y) = b$ such that $a < b$ and $0 < \tilde{c}(x) - \tilde{c}(y) < n$. Denote by $\mathrm{inv}(S)$ the number of inversions in S.

Lemma 5.2.2. *Given a (skew) shape μ, there is a constant e such that for every standard n-ribbon tableau T of shape μ, we have* $\mathrm{sp}(T) = e - \mathrm{inv}(\mathrm{quot}_n(T))$.

Proof. Say that standard tableaux S, S' of shape $\boldsymbol{\mu} = \mathrm{quot}_n(\mu)$ differ by a *switch* if they are identical except for the positions of two consecutive entries a, $a + 1$. Every pair of tableaux is connected by a sequence of switches. If S and S' differ by a switch, then the only entries that might form an inversion in one tableau but not the other are a, $a + 1$. Assuming as we may that $\mathrm{inv}(S') \geq \mathrm{inv}(S)$, we therefore have $\mathrm{inv}(S') = \mathrm{inv}(S) + \epsilon$ with $\epsilon \in \{0, 1\}$.

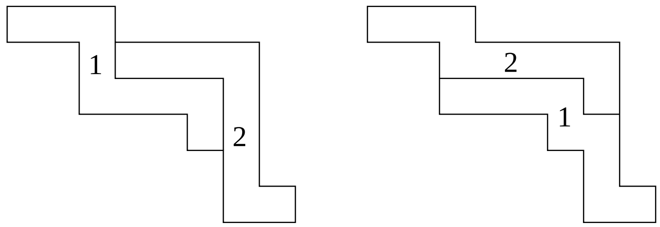

FIGURE 4. A shape with two tilings by two n-ribbons

Let $S = \mathrm{quot}_n(T)$, $S' = \mathrm{quot}_n(T')$. What must be shown is that $\mathrm{sp}(T) = \mathrm{sp}(T') + \epsilon$. Since T and T' are identical except for the ribbons labelled a and $a+1$, the problem reduces to the case $|\mu| = 2n$, $|\boldsymbol{\mu}| = 2$. Let x, y be the two cells of $\boldsymbol{\mu}$, with $S(x) = 1$, $S(y) = 2$ and $S'(x) = 2$, $S'(y) = 1$. Note that $\tilde{c}(x) \neq \tilde{c}(y)$. We have $\epsilon = 0$ if and only if $|\tilde{c}(y) - \tilde{c}(x)| \geq n$. This means that μ is the union of two ribbons whose cells have no contents in common, and these ribbons are unique. Hence $\mathrm{sp}(T) = \mathrm{sp}(T')$.

Conversely, we have $\epsilon = 1$ if and only if $|\tilde{c}(y) - \tilde{c}(x)| < n$, which means that μ can be tiled by two ribbons whose cells have at least one content in common. In this case μ has exactly two ribbon tilings, as in Figure 4, each supporting one of the standard tableaux T, T'. In fact, μ is both a horizontal and a vertical n-ribbon strip, the tiling of T is the official tiling of μ as a horizontal strip, and that of T' is the official tiling as a vertical strip. In the horizontal tiling, each ribbon has one more row than the corresponding ribbon with the same content in the vertical tiling (see [25, Lemma 4.1]). Hence $\mathrm{sp}(T) = \mathrm{sp}(T') + 1$. $\square$

Lemma 5.2.3. *Lemma 5.2.2 holds also for semistandard tableaux.*

Proof. If S is a standard tableau of shape $\boldsymbol{\mu} = \mathrm{quot}_n(\mu)$, call a a *descent* of S if $S(x) = a$, $S(y) = a+1$, with $\tilde{c}(x) > \tilde{c}(y)$. If $\boldsymbol{\mu}$ is a horizontal strip, there is a unique standard tableau of shape $\boldsymbol{\mu}$ with no descents, and conversely, such a tableau exists only on a horizontal strip. As in the proof of Proposition 2.4.2, it follows that each semistandard tableau T has a unique standardization S such that $T \circ S^{-1}$ is weakly increasing, and if $T \circ S^{-1}(j) = T \circ S^{-1}(j+1) = \cdots = T \circ S^{-1}(k)$, then $d(T) \cap \{j, \ldots, k-1\} = \emptyset$.

By definition, equal entries $T(x) = T(y) = a$ contribute nothing to $\mathrm{inv}(T)$. In the standardization S, equal entries are replaced with entries labelled in increasing order of adjusted content $\tilde{c}$, which contribute nothing to $\mathrm{inv}(S)$. On the other hand, unequal entries of T give rise to entries ordered in the same way in S, so $\mathrm{inv}(T) = \mathrm{inv}(S)$.

Given a semistandard n-ribbon tableau T, we define its standardization to be the unique standard n-ribbon tableau S such that $\mathrm{quot}_n(S)$ is the standardization of $\mathrm{quot}_n(T)$, as above. The important point to notice here is that for each letter a, the horizontal ribbon strip $T^{-1}(\{a\})$ has its ribbons labelled in S in increasing order of content, and hence it is tiled in S by the official tiling. This shows that T and its standardization have the same underlying ribbon tiling, and hence the same spin $\mathrm{sp}(T) = \mathrm{sp}(S)$.

These observations reduce the lemma for semistandard tableaux to the standard case. $\square$

Remark. Schilling, Shimozono and White [26] defined an inversion number $\mathrm{inv}'(T)$ such that $\mathrm{csp}(T) = \mathrm{inv}'(\mathrm{quot}(T))$ exactly, without the constant error term e in Lemma 5.2.2. An inversion by their definition is an inversion by ours which also satisfies some extra conditions. Haiman's student Michelle Bylund and Haiman found the simpler definition used here and the proof given above.

Corollary 5.2.4. *Fix an n-tuple of shapes $\boldsymbol{\mu} = (\mu^{(0)}, \dots, \mu^{(n-1)})$ and a sequence of content offsets $s_i \equiv i \pmod{n}$ as in the definitions of $\tilde{c}$ and* inv*. Then there exists μ with $\mathrm{quot}_n(\mu) = \boldsymbol{\mu}$ such that*

$$q^e G_\mu(z; q^{-1}) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\mu})} q^{\mathrm{inv}(T)} z^T \tag{76}$$

for some exponent e. In particular, the expression on the right-hand side is a symmetric function.

Proof. This is immediate from Lemma 5.2.3 if $(s_0, s_1, \dots, s_{n-1})$ is the content sequence of some n-core ν. But we can always make it one by adding $c(n, n, \dots, n)$ for some c. This change does not alter the value of $\mathrm{inv}(T)$. $\square$

Proof of Theorem 3.1.3 (symmetry). We are to show that the expression $D_n^\lambda(z; q)$ in (32) is a symmetric function. Write $\lambda = (0^{\alpha_0}, 1^{\alpha_1}, \dots, (n-1)^{\alpha_{n-1}})$, defining α_0 so that $\sum_j \alpha_j = n$. Set $\mu^{(j)} = (1^{\alpha_j})$, a single column of the same height as column j in $(\lambda + (1^n))/\lambda$. Translating the columns of $(\lambda + (1^n))/\lambda$ onto the corresponding columns $\mu^{(j)}$ gives a bijection between the cells of $(\lambda + (1^n))/\lambda$ and those of $\boldsymbol{\mu} = (\mu^{(0)}, \dots, \mu^{(n-1)})$. This induces in the obvious way a bijection between semistandard tableaux of these shapes.

Take the offsets $s_j = j - n(j + \lambda'_{j+1})$. Then if $x \in (\lambda + (1^n))/\lambda$ corresponds to $x' \in \mu^{(j)}$, we have

$$\tilde{c}(x') = j - n d(x). \tag{77}$$

It follows that conditions (i) or (ii) in the definition of d-inversion for cells $x, y \in (\lambda + (1^n))/\lambda$ are equivalent to the condition $0 < \tilde{c}(x') - \tilde{c}(y') < n$ for the corresponding cells $x', y' \in \boldsymbol{\mu}$. If $T \in \mathrm{SSYT}(\lambda + (1^n)/\lambda)$ corresponds to $T' \in \mathrm{SSYT}(\boldsymbol{\mu})$ we therefore have $\mathrm{dinv}(T) = \mathrm{inv}(T')$. Hence $D_n^\lambda(z; q)$ coincides with the right-hand side of (76) for this choice of $\boldsymbol{\mu}$ and s_i. $\square$

Example. We make the constructions in the proof more explicit for the tableau T shown in Figure 2. For this T, we have $\boldsymbol{\mu} = ((1^2), (1^2), \emptyset, (1), \emptyset, (1), (1^2), \emptyset)$ and

$$T' = \left(\begin{array}{|c|}\hline 8 \\ \hline 6 \\ \hline\end{array}, \begin{array}{|c|}\hline 7 \\ \hline 5 \\ \hline\end{array}, \emptyset, \begin{array}{|c|}\hline 1 \\ \hline\end{array}, \emptyset, \begin{array}{|c|}\hline 2 \\ \hline\end{array}, \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline\end{array}, \emptyset \right).$$

The content offsets are $(-48, -39, -46, -45, -52, -51, -42, -49)$, giving adjusted contents

$$\left(\begin{array}{|c|}\hline -56 \\ \hline -48 \\ \hline\end{array}, \begin{array}{|c|}\hline -47 \\ \hline -39 \\ \hline\end{array}, \emptyset, \begin{array}{|c|}\hline -45 \\ \hline\end{array}, \emptyset, \begin{array}{|c|}\hline -51 \\ \hline\end{array}, \begin{array}{|c|}\hline -50 \\ \hline -42 \\ \hline\end{array}, \emptyset \right).$$

The reader can verify that each d-inversion in T listed after eq. (28) corresponds to a pair of entries $T'(x') < T'(y')$ in cells x', y' with adjusted contents $0 < \tilde{c}(x') - \tilde{c}(y') < 8$. For example, the d-inversion $(4, 1)$ has $\tilde{c}(x') = -45$, $\tilde{c}(y') = -50$.

5.3. **Positivity.** When μ is a partition with $\mathrm{core}_n(\mu) = \emptyset$, Leclerc and Thibon [20] have shown that the coefficient

$$\langle q^{\mathrm{smin}(\mu)} G_\mu(z; q^2), s_\lambda(z)\rangle \tag{78}$$

(which is a q-analog of the Littlewood-Richardson coefficient $c^\lambda_{\mu^{(0)},\ldots,\mu^{(n-1)}}$) coincides with a parabolic Kazhdan-Lusztig polynomial $P^-_{\mu+\rho,n\lambda+\rho}(q)$ for a suitable affine symmetric group $\widehat{S}_r$. In turn, Kashiwara and Tanisaki [16] have interpreted the coefficients of these polynomials as decomposition multiplicities for certain non-irreducible perverse sheaves on affine partial flag varieties, which shows that they are positive. This together with our work in §5.2 would immediately imply the positivity part of Theorem 3.1.3, were it not for the fact that we had to introduce non-trivial offsets s_i. To account for them and complete the proof of Theorem 3.1.3 we need the following small improvement on the results of [20].

Proposition 5.3.1. *Let μ be a partition and set $\nu = \mathrm{core}_n(\mu)$. Then*

$$\langle q^{\mathrm{smin}(\mu/\nu)} G_{\mu/\nu}(z; q^2), s_\lambda(z)\rangle = P^-_{\mu+\rho,\nu+n\lambda+\rho}(q) \tag{79}$$

is a parabolic Kazhdan-Lusztig polynomial—written here using the notation of [20].

Since this is essentially a result of Leclerc and Thibon, we will confine ourselves to brief remarks on what is needed to deduce it from the contents of [20]. Leclerc and Thibon work in a q-Fock space $\mathcal{F}_r$, where r is arbitrary, provided it is greater than or equal to the length of all partitions under discussion. The space $\mathcal{F}_r$ is equipped with a natural basis, whose elements are denoted $|\mu + \rho\rangle$ (with $\rho = \delta_r$), and a Kazhdan-Lusztig type basis, denoted $G^-_{\mu+\rho}$. The Kazhdan-Lusztig polynomials $P^-_{\lambda+\rho,\mu+\rho}(-q^{-1})$ are the coefficients of $G^-_{\mu+\rho}$ with respect to the natural basis. The use of $-q^{-1}$ is an artifact of the notation; it gets replaced by q in the end. The algebra of symmetric functions in r variables acts on $\mathcal{F}_r$ in such a way that $\langle (-q)^{-\mathrm{smin}(\mu/\nu)} G_{\mu/\nu}(z; q^{-2}), s_\lambda(z)\rangle$ is the coefficient of $|\mu + \rho\rangle$ in $s_\lambda \cdot |\nu + \rho\rangle$.

A partition ν is called *n-restricted* if $\nu_i - \nu_{i+1} < n$ for all i. Leclerc and Thibon prove that if ν is n-restricted, then $s_\lambda \cdot G^-_{\nu+\rho} = G^-_{\nu+n\lambda+\rho}$. Proposition 5.3.1 follows immediately from the foregoing and the following two additional facts.

Lemma 5.3.2. *Any n-core is n-restricted.*

Proof. Obvious. □

Lemma 5.3.3. *If ν is an n-core, then $G^-_{\nu+\rho} = |\nu + \rho\rangle$.*

Proof. Using the notation of [20], it suffices to prove that $\overline{|\nu + \rho\rangle} = |\nu + \rho\rangle$. By [20, Prop 5.9], there are well-defined elements $|\gamma\rangle \in \mathcal{F}_r$ for any sequence $\gamma = (\gamma_1, \ldots, \gamma_r)$, not necessarily

decreasing, satisfying the following straightening relations when $\gamma_{i+1} - \gamma_i = kn + j$ with $k \geq 0$ and $0 \leq j < n$.

$$|\gamma\rangle = 0 \quad \text{if } j = k = 0, \tag{80}$$

$$|\gamma\rangle = -|\sigma_i\gamma\rangle \quad \text{if } j = 0 \text{ and } k \neq 0, \tag{81}$$

$$|\gamma\rangle = -q^{-1}|\sigma_i\gamma\rangle \quad \text{if } k = 0 \text{ and } j \neq 0, \tag{82}$$

$$|\gamma\rangle = -q^{-1}|\sigma_i\gamma\rangle - |y_i^{-k}y_{i+1}^{k}\gamma\rangle - q^{-1}|y_i^{k}y_{i+1}^{-k}\sigma_i\gamma\rangle \quad \text{otherwise.} \tag{83}$$

Here σ_i is the transposition exchanging γ_i and γ_{i+1} and y_i is the operator that adds n to γ_i. Let w_0 denote the longest permutation in S_r. By [20, Prop. 5.7 and Cor. 5.10], we have

$$\overline{|\nu + \rho\rangle} = (-1)^{l(w_0)}q^e|w_0(\nu + \rho)\rangle = |\nu + \rho\rangle + \sum_{\lambda} a_{\lambda+\rho,\nu+\rho}(q)|\lambda + \rho\rangle \tag{84}$$

for a suitable exponent e, and all terms $|\lambda + \rho\rangle$ in the sum on the right, which arise from the process of straightening $|w_0(\nu + \rho)\rangle$, are lexicographically less than $\nu + \rho$. From the straightening relations we see that all these terms have $|\lambda| = |\nu|$, and the multiset of congruence classes $\lambda_i + \rho_i \pmod n$ is the same as that of $\nu + \rho$. However, this last condition implies that $\mathrm{core}_n(\lambda) = \mathrm{core}_n(\nu) = \nu$, which is absurd, since $|\lambda| = |\nu|$ and $\lambda \neq \nu$. In short, the final sum on the right in (84) vanishes, yielding the desired result. □

Remark. It is conjectured that $G_{\mu/\nu}(z; q)$ is Schur positive even when ν is not an n-core. One can use the results of [20] to write $\langle q^{\mathrm{smin}(\mu/\nu)}G_{\mu/\nu}(z; q^2), s_\lambda(z)\rangle$ explicitly in terms of Kazhdan-Lusztig polynomials, but in general the resulting expressions contain negative terms.

Proof of Theorem 3.1.3 (positivity). In the proof of the symmetry part of the theorem, we have shown that $D_n^\lambda(z; q) = q^e G_\mu(z; q^{-1})$ for a suitable exponent e and skew shape μ. For this μ, each $\mu^{(i)} = (1^{\alpha_i})$ is a partition shape, so $\mu = \eta/\nu$, where $\nu = \mathrm{core}_n(\eta)$ is the n-core associated to the specified offsets s_i. Hence $G_\mu(z; q)$ is Schur positive by Proposition 5.3.1. □

6. Higher powers

In this section we explore to what extent the preceding conjectures and results generalize to the higher powers $\nabla^m e_n(z)$.

6.1. **The meaning of $\nabla^m e_n(z)$.** As explained in the introduction, $\nabla e_n(z)$ is the Frobenius series of the diagonal coinvariant ring R_n. The higher powers $\nabla^m e_n(z)$ have a similar interpretation, which shows that they are also Schur positive.

Proposition 6.1.1. *Let $I = ((\mathbf{x}, \mathbf{y}) \cap \mathbb{C}[\mathbf{x}, \mathbf{y}]^{S_n})$ be the ideal in $\mathbb{C}[\mathbf{x}, \mathbf{y}]$ generated by all S_n-invariant polynomials without constant term, and $J = (\mathbb{C}[\mathbf{x}, \mathbf{y}]^\epsilon)$ the ideal generated by all antisymmetric polynomials. Then*

$$\nabla^m e_n(z) = \mathcal{F}_{\varepsilon^{m-1}\otimes J^{m-1}/IJ^{m-1}}(z; q, t) \tag{85}$$

where ε is the sign representation.

Proof. Although this follows from the methods of [13], it wasn't shown explicitly there, so we explain what more needs to be said. As in [13, Cor. 3.5, eq. (110)], the quantity $\nabla^m e_n(z)$ coincides with [13, Theorem 3.3, eq. (107)], with the factor $s_\nu[B_\mu(q,t)]$ there replaced by $e_n^{m-1}[B_\mu(q,t)] = t^{(m-1)n(\mu)}q^{(m-1)n(\mu')}$. It follows from that theorem that $\nabla^m e_n(z)$ is the Frobenius series of

$$(R(n,(m-1)n)/\mathfrak{m}R(n,(m-1)n))^\epsilon, \tag{86}$$

where $l = (m-1)n$, $R(n,l)$ and $\mathfrak{m}$ are as in [13], and $(-)^\epsilon$ denotes the space of antisymmetric elements with respect to the action of $S_n^{m-1} \subseteq S_l$. The proof of [12, Prop. 4.11.1] identifies $R(n,(m-1)n)^\epsilon$ with $\varepsilon^{m-1} \otimes J^{m-1}$, and with this identification, (86) becomes $\varepsilon^{m-1} \otimes J^{m-1}/IJ^{m-1}$ in our present notation.

□

6.2. **An extension of Conjecture 3.1.2.** We begin by generalizing the notion of d-inversion and the statistic $\mathrm{dinv}(T)$. Our new definitions depend on the integer m and reduce for $m = 1$ to the definitions of d-inversion and $\mathrm{dinv}(T)$ in §3.1.

For each cell $x = (i,j) \in \mathbb{N} \times \mathbb{N}$, put $d_m(x) = mi + j$ (this keeps track of which diagonal of slope $-1/m$ contains x). Given $\lambda \subseteq m\delta_n$, let T be a semistandard tableau of shape $(\lambda + (1^n))/\lambda$. Let $T(x) = a$, $T(y) = b$ be two entries with $a < b$, and put $x = (i,j)$, $y = (i',j')$. We say that this pair of entries contributes anywhere from 0 to m *d-inversions*, according to the following rules:

(i) if $j > j'$, this pair contributes $\max(0, m - |d_m(y) - d_m(x)|)$ inversions;
(ii) if $j < j'$, it contributes $\max(0, m - |d_m(y) - d_m(x) - 1|)$ inversions.

We also allow equal entries to contribute. Define the reverse diagonal lexicographic order by

$$x <_d y \quad \text{if } d_m(x) > d_m(y), \text{ or } d_m(x) = d_m(y) \text{ and } j < j', \text{ where } x = (i,j),\ y = (i',j').$$

Then a pair of equal entries $T(x) = T(y) = a$ contributes the same number of d-inversions as would a pair of unequal entries $T(x) = a$, $T(y) = b$ with $x <_d y$ and $a < b$.

We extend these definitions to super tableaux by applying the rules above for unequal entries $T(x) = a$, $T(y) = b$ and for equal entries $T(x) = T(y) = a \in \mathcal{A}_+$. For negative entries, we use the opposite rule: a pair of entries $T(x) = T(y) = a \in \mathcal{A}_-$ contributes the same number of d-inversions as would a pair of unequal entries $T(x) = a$, $T(y) = b$ with $x >_d y$ and $a < b$.

Remark. Another way to formulate the rule for a pair of equal entries $T(x) = T(y) = a$ is that if a is positive, it contributes the minimum of the two alternatives described by (i) and (ii) above; if a is negative, it contributes the maximum.

Let $\mathrm{dinv}_m(T)$ denote the total number of d-inversions contributed by pairs of entries in T. The extensions of Definition 3.1.1 and Conjecture 3.1.2 are as follows.

Definition 6.2.1.

$$D_n^{(m)}(z;q,t) = \sum_{\lambda \subseteq m\delta_n} \sum_{T \in \mathrm{SSYT}(\lambda+(1^n)/\lambda)} t^{|m\delta_n/\lambda|} q^{\mathrm{dinv}_m(T)} z^T. \tag{87}$$

Conjecture 6.2.2. *We have the identity*

$$\nabla^m e_n(z) = D_n^{(m)}(z; q, t). \tag{88}$$

Equivalently, for all μ we have

$$\langle \nabla^m e_n, h_\mu \rangle = \sum_{\lambda \subseteq m\delta_n} \sum_{T \in \mathrm{SSYT}(\lambda + (1^n)/\lambda,\, \mu)} t^{|m\delta_n/\lambda|} q^{\mathrm{dinv}_m(T)}. \tag{89}$$

Theorem 3.1.3 now generalizes just as we should expect.

Theorem 6.2.3. *The quantity $D_n^{(m)}(z; q, t)$ is a symmetric function in z, and it is Schur positive. In fact, each term*

$$D_n^{(m),\lambda}(z; q) = \sum_{T \in \mathrm{SSYT}(\lambda + (1^n)/\lambda)} q^{\mathrm{dinv}_m(T)} z^T. \tag{90}$$

is individually symmetric and Schur positive.

Theorem 6.2.3 will be proven in §6.4. Granting it for the moment, let us deduce some consequences.

Theorem 6.2.4. *The superization $\tilde{D}_n^{(m)}(z, w; q, t) = \omega^W D_n^{(m)}[Z + W; q, t]$ is given by*

$$\tilde{D}_n^{(m)}(z, w; q, t) = \sum_{\lambda \subseteq m\delta_n} \sum_{T \in \mathrm{SSYT}_\pm(\lambda + (1^n)/\lambda)} t^{|m\delta_n/\lambda|} q^{\mathrm{dinv}_m(T)} z^T. \tag{91}$$

Equivalently, Conjecture 6.2.2 implies

$$\langle \nabla^m e_n, e_\eta h_\mu \rangle = \sum_{\lambda \subseteq m\delta_n} \sum_{T \in \mathrm{SSYT}_\pm(\lambda + (1^n)/\lambda,\, \mu,\, \eta)} t^{|\delta_n/\lambda|} q^{\mathrm{dinv}_m(T)}. \tag{92}$$

Proof. The proof of Theorem 3.2.1 applies almost verbatim. Only the verification that a super tableau T and its standardization S satisfy $\mathrm{dinv}(S) = \mathrm{dinv}(T)$ needs to be adapted to the case of general m. But this is immediate, given our rule for the number of d-inversions contributed by a pair of equal entries. $\square$

6.3. **Specializations.** We examine the analogs for $D_n^{(m)}(z; q, t)$ of some of the specializations of $D_n(z; q, t)$ described in §4. Beginning with the easiest specialization, at $q = 1$, we have by [5, Thms. 5.2, 5.3] the following analog of (40):

$$\nabla^m e_n(z)|_{q=1} = \sum_{\lambda \subseteq m\delta_n} t^{|m\delta_n/\lambda|} e_\alpha(z), \tag{93}$$

where $\lambda = (0^{\alpha_0}, 1^{\alpha_1}, 2^{\alpha_2}, \ldots)$ and $\sum_i \alpha_i = n$. Clearly, this coincides with $D_n^{(m)}(z; 1, t)$.

For the specializations $t = 0$ and $q = 0$, we first observe that the analog of Lemma 4.2.1 is the identity

$$\nabla^m e_n(z)|_{t=0} = q^{(m-1)\binom{n}{2}} (q; q)_n h_n[Z/(1 - q)]. \tag{94}$$

This can be deduced from Lemma 4.2.1 either by observing that the right-hand side of (41) is the modified Macdonald polynomial $\tilde{H}_{(n)}$, or by using Proposition 6.1.1 and the fact that $J^{m-1} \cap \mathbb{C}[\mathbf{x}]$ is the principal ideal generated by $\Delta(\mathbf{x})^{m-1}$, where $\Delta(\mathbf{x})$ is the Vandermonde determinant in the variables $\mathbf{x} = x_1, \ldots, x_n$. Multiplication by $\Delta(\mathbf{x})^{m-1}$ induces an isomorphism $R_n \cap \mathbb{C}[\mathbf{x}] \to (J^{m-1} \cap \mathbb{C}[\mathbf{x}])/(IJ^{m-1} \cap \mathbb{C}[\mathbf{x}])$ which is homogeneous of degree $(m-1)\binom{n}{2}$.

Proposition 6.3.1. *We have*

$$D_n^{(m)}(z; q, 0) = \nabla^m e_n(z)|_{t=0}. \tag{95}$$

Proof. Same as the proof of Proposition 4.2.2 except that now the sole term is $\lambda = m\delta_n$, and $\operatorname{dinv}_m(T)$ is the number of ordinary inversions plus $(m-1)\binom{n}{2}$. $\square$

Proposition 6.3.2. *We have*

$$D_n^{(m)}(z; 0, t) = \nabla^m e_n(z)|_{q=0}. \tag{96}$$

Proof. One can verify that the criterion in Lemma 4.3.1 for a standard tableau $T \in \mathrm{SYT}(\lambda + (1^n)/\lambda)$ to have $\operatorname{dinv}_m(T) = 0$ is the same for general m as it is for $m = 1$. The only difference for $m > 1$ is that $|m\delta_n/\lambda| = |\delta_n/\lambda| + (m-1)\binom{n}{2}$, which shows that $D_n^{(m)}(z; 0, t) = t^{(m-1)\binom{n}{2}} D_n(z; 0, t)$. Exchanging q and t in (94), and using Proposition 4.3.2, we see that this agrees with (96). $\square$

Next we turn to the Catalan specialization. Higher q, t-Catalan polynomials were defined in [5] (see also [11]) by a formula which amounts to

$$C_n^{(m)}(q, t) = \langle \nabla^m e_n, e_n \rangle. \tag{97}$$

From (92), we see that Conjecture 6.2.2 implies the following analog of (51).

$$C_n^{(m)}(q, t) = \sum_{\lambda \subseteq m\delta_n} t^{|m\delta_n/\lambda|} q^{\operatorname{dinv}_m(\lambda)}. \tag{98}$$

The analog of (52) is a conjecture of Haiman (see also [22]) that

$$C_n^{(m)}(q, t) = \sum_{\lambda \subseteq m\delta_n} t^{|m\delta_n/\lambda|} q^{b_m(\lambda)}, \tag{99}$$

where $b_m(\lambda)$ is the number of cells $x \in \lambda$ such that

$$ml(x) \leq a(x) \leq ml(x) + m. \tag{100}$$

We will verify that (98) and (99) are equivalent (but not that they are true—see §7, Problem 2).

Lemma 6.3.3. *We have $b_m(\lambda) = \operatorname{dinv}_m(T)$, where T is the super tableau of shape $(\lambda + (1^n))/\lambda$ whose every entry is the negative letter $\bar{1}$.*

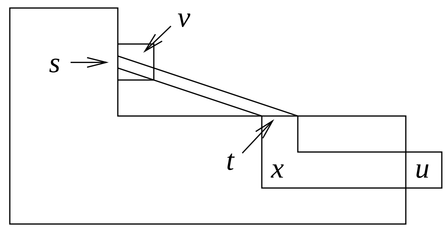

FIGURE 5. Construction of the cells u, v corresponding to x.

Proof. Let $\mathcal{D}$ be the lattice path from $(n, 0)$ to $(0, mn)$ formed by the outer boundary of λ together with segments of the i and j-axes. When $m = 1$, this is the Dyck path associated with λ. In the general case, it is a lattice path that never goes above the line $i + j/m = n$.

Given a cell $x \in \lambda$, let u be the cell of $(\lambda + (1^n))/\lambda$ just outside the end of the arm of x. Let t be the unit-width segment at the end of the leg of x. Project t along diagonals of slope $-1/m$ onto a vertical segment s of $\mathcal{D}$ of height $1/m$, as indicated in Figure 5, and let v be the cell of $(\lambda + (1^n))/\lambda$ containing s on its left border. Note that the point of $\mathbb{N} \times \mathbb{N}$ represented by each cell is its lower-left corner.

If s_0 is the lower end of the segment s and t_0 is the left end of t, then we have

$$d_m(s_0) = d_m(t_0) = d_m(x) + ml(x) + m \tag{101}$$

$$d_m(u) = d_m(x) + a(x) + 1, \tag{102}$$

whence $a(x) - ml(x) = d_m(u) - d_m(s_0) + m - 1$. If x satisfies (100), we therefore have

$$d_m(u) - 1 \leq d_m(s_0) \leq d_m(u) + m - 1. \tag{103}$$

Conversely, given $u \in (\lambda + (1^n))/\lambda$ and a vertical segment s of $\mathcal{D}$ of height $1/m$, situated to the left of u and with endpoints in $(\frac{1}{m}\mathbb{N}) \times \mathbb{N}$, the inequality $d_m(s_0) \leq d_m(u) + m - 1$ implies that the diagonal through the upper endpoint s_1 lies on or to the left of the upper-left corner of u. Hence we can project s onto a unit-width horizontal segment t of $\mathcal{D}$ with t and u belonging to the hook of a (unique) cell $x \in \lambda$, as in the construction above. Condition (100) for this x is then equivalent to (103).

Now, given $u, v \in (\lambda + (1^n))/\lambda$ with v to the left of u, the number of segments s as above that lie on the left border of v and satisfy (103) is equal to

$$\min(d_m(v) + m - 1, d_m(u) + m - 1) - \max(d_m(v), d_m(u) - 1) + 1, \tag{104}$$

or to zero if this expression is negative. If $d_m(u) > d_m(v)$ this simplifies to $\max(0, m - (d_m(u) - d_m(v) - 1))$. Otherwise it is $\max(0, m - (d_m(v) - d_m(u)))$.

In T, the cells u and v contain equal negative entries. If $d_m(u) > d_m(v)$ then $u <_d v$ and this pair contributes $\max(0, m - |d_m(u) - d_m(v) - 1|) = \max(0, m - (d_m(u) - d_m(v) - 1))$ d-inversions. Otherwise, $u >_d v$ and it contributes $\max(0, m - |d_m(v) - d_m(u)|) = \max(0, m - (d_m(v) - d_m(u)))$ d-inversions. The lemma is proved. $\square$

The analog for $m > 1$ of the Haglund-Loehr conjecture discussed in §4.5 is a conjecture of Loehr and Remmel [23] for the value of $\langle \nabla^m e_n, e_1^n \rangle$. It is a simple observation that their conjecture is equivalent to $\langle \nabla^m e_n, e_1^n \rangle = \langle D_n^{(m)}(z; q, t), e_1^n \rangle$. In this connection we should mention that Loehr [22] has given a fermionic formula for the quantity here denoted $\langle D_n^{(m)}(z; q, t), e_1^n \rangle$, and also for similar quantities in which $D_n^{(m)}(z; q, t)$ is replaced by a sum over partitions λ contained in a more general trapezoidal shape $(l^n) + m\delta_n$.

Finally, we expect the analog for $m > 1$ of Conjecture 4.7.1 to be the following.

Conjecture 6.3.4. *For $1 \leq k \leq n$,*

$$\nabla^m E_{n,k} = \sum_{\substack{\lambda \subseteq m\delta_n \\ |\{i : \lambda_i = m(n-i)\}| = k}} t^{|m\delta_n/\lambda|} D_n^{(m),\lambda}(z; q). \tag{105}$$

6.4. **Proof of Theorem 6.2.3.** In this section, d-inversions and $<_d$ are always defined with respect to the given integer m. We begin with a lemma.

Lemma 6.4.1. *Let u, v be cells in different columns of a tableau T, with v to the left of u.*

(a) If $1 - m \leq d_m(u) - d_m(v) \leq 0$, then entries $T(u) < T(v)$ contribute one more d-inversion than entries $T(u) > T(v)$.

(b) If $1 \leq d_m(u) - d_m(v) \leq m$, then entries $T(u) > T(v)$ contribute one more d-inversion than entries $T(u) < T(v)$.

(c) Otherwise, the number of d-inversions contributed by entries $T(u)$, $T(v)$ is zero in any case.

Proof. Referring to the rule in §6.2 for the number of d-inversions contributed, we see that case (i) occurs when $T(u) < T(v)$, and contributes $\max(0, m - |d_m(v) - d_m(u)|) = \max(0, m - |d_m(u) - d_m(v)|)$ d-inversions. Otherwise, case (ii) occurs and contributes $\max(0, m - |d_m(u) - d_m(v) - 1|)$ d-inversions.

If $d_m(u) > d_m(v)$ then both $d_m(u) - d_m(v)$ and $d_m(u) - d_m(v) - 1$ are non-negative. Then case (ii) contributes one more d-inversion than case (i), unless $d_m(u) - d_m(v) > m$, in which event both cases contribute zero d-inversions. If $d_m(u) \leq d_m(v)$ then both $d_m(u) - d_m(v)$ and $d_m(u) - d_m(v) - 1$ are non-positive. Then case (i) contributes one more d-inversion than case (ii), unless $d_m(u) - d_m(v) \leq -m$, in which event there are again zero d-inversions in either case. □

Using this lemma, we can simplify the rule for counting d-inversions, at the price of adding an overall constant. Let T be semistandard of shape $(\lambda + (1^n))/\lambda$. We say that entries $T(x) = a$, $T(y) = b$, with $a < b$ and $x = (i, j)$, $y = (i', j')$ contribute a *reduced d-inversion* if either

(i)′ $0 \leq d_m(y) - d_m(x) \leq m - 1$ and $j > j'$, or

(ii)′ $1 \leq d_m(y) - d_m(x) \leq m$ and $j < j'$.

Pairs of equal (positive) entries do not contribute any reduced d-inversions. Note that this agrees with the rule that equal entries count as if they were unequal entries $a < b$ with $x <_d y$, since both (i)′ and (ii)′ imply $x >_d y$.

Let $\mathrm{dinv}'_m(T)$ denote the number of reduced d-inversions in T. Then we can rephrase Lemma 6.4.1 as follows.

Corollary 6.4.2. *There is a constant $e(\nu)$ depending only on $\nu = (\lambda + (1^n))/\lambda$ such that $\mathrm{dinv}_m(T) = e(\nu) + \mathrm{dinv}'_m(T)$ for all $T \in \mathrm{SSYT}(\nu)$. Consequently,*

$$D_n^{(m),\lambda}(z;q) = q^{e(\nu)} \sum_{T \in \mathrm{SSYT}(\lambda+(1^n)/\lambda)} q^{\mathrm{dinv}'_m(T)} z^T. \tag{106}$$

Remark. In effect, the constant $e(\nu)$ is $\mathrm{dinv}_m(T_0)$, where T_0 is the tableau of shape ν whose entries are all 1 (however, T_0 is generally not a legal semistandard tableau).

To prove Theorem 6.2.3, it suffices to identify the sum on the right-hand side of (106) with an expression of the form (76), in this case with n replaced by $mn+1$, so $\boldsymbol{\mu} = (\mu^{(0)}, \dots, \mu^{(mn)})$, and each $\mu^{(j)}$ a partition shape in order to deduce positivity along with symmetry.

To this end, write $\lambda = (0^{\alpha_0}, 1^{\alpha_1}, \dots, mn^{\alpha_{mn}})$ with α_0 defined so that $\sum_j \alpha_j = n$. Let β be the permutation of $\{0, 1, \dots, mn\}$ such that

$$\beta(j) \equiv -nj \pmod{mn+1}. \tag{107}$$

It exists because n is relatively prime to $mn+1$. Note that for $m = 1$, β is the identity permutation, which is why it did not come up earlier in the proof for that case. Define each $\mu^{(j)}$ to be a single column, such that

$$\mu^{(\beta(j))} = (1^{\alpha_j}). \tag{108}$$

We have a natural bijection between cells $x \in (\lambda + (1^n))/\lambda$ and $x' \in \boldsymbol{\mu}$, which translates column j of $(\lambda + (1^n))/\lambda$ onto $\mu^{(\beta(j))}$. This induces a bijection of semistandard tableaux in the obvious way.

Define content offsets

$$s_{\beta(j)} = -nj - (mn+1)\lambda'_{j+1}. \tag{109}$$

Note that the right hand side is congruent to $\beta(j) \pmod{mn+1}$, as it should be. With these offsets, the adjusted content of the cell $x' \in \boldsymbol{\mu}$ corresponding to $x = (i,j) \in (\lambda + (1^n))/\lambda$ is

$$\tilde{c}(x') = -(mn+1)i - nj = -i - n d_m(x). \tag{110}$$

For any two distinct cells $x = (i,j)$ and $y = (i',j')$, we have $0 < |i - i'| < n$. It follows that the inequalities

$$0 < \tilde{c}(x') - \tilde{c}(y') < mn+1 \tag{111}$$

hold if and only if

$$0 \le d_m(y) - d_m(x) \le m, \tag{112}$$

and also $i < i'$ if $d_m(y) - d_m(x) = 0$, and $i > i'$ if $d_m(y) - d_m(x) = m$. Moreover, since $j = j'$ implies that $mn+1$ divides $\tilde{c}(y) - \tilde{c}(x)$, these conditions imply $j \neq j'$. Hence we have $i < i' \Leftrightarrow j > j'$ and $i > i' \Leftrightarrow j < j'$. In short, inequalities (111) hold if and only if (i)$'$ or (ii)$'$ holds for the cells x, y.

This shows that if $S \in \mathrm{SSYT}(\boldsymbol{\mu})$ corresponds under the natural bijection to $T \in \mathrm{SSYT}(\lambda+(1^n)/\lambda)$, then $\mathrm{inv}(S) = \mathrm{dinv}'_m(T)$. The desired result follows. □

7. Open problems

Problem 1. Prove Conjecture 3.1.2 and, more generally, Conjecture 6.2.2.

Problem 2. Prove that $\langle \nabla^m e_n, e_n \rangle = \langle D_n^{(m)}(z;q,t), e_n \rangle$ for $m > 1$, or what is the same, prove the combinatorial formulae (98) and (99) for $C_n^{(m)}(q,t)$. This problem might be amenable to attack by methods similar to those used in [3, 4] for the case $m = 1$.

Problem 3. Prove that $D_n(z;q,t) = D_n(z;t,q)$, and similarly for $D_n^{(m)}(z;q,t)$. This would remain a relevant combinatorial problem even if the conjectures were proven. By way of illustration, although the combinatorial formula (51) for $C_n(q,t)$ has been proven, no combinatorial interpretation of the symmetry $C_n(q,t) = C_n(t,q)$ is known at present.

Problem 4. Prove that $D_n^{(m)}(z;q,1) = D_n^{(m)}(z;1,q)$. This specialization of Problem 3 should be easier than the full q,t symmetry. Combinatorial interpretations are known [2, 21, 22, 23] for the special cases corresponding to the identities $C_n^{(m)}(q,1) = C_n^{(m)}(1,q)$, $\langle D_n(z;q,1), e_{n-d}h_d \rangle = \langle D_n(z;1,q), e_{n-d}h_d \rangle$ and $\langle D_n^{(m)}(z;q,1), e_1^n \rangle = \langle D_n^{(m)}(z;1,q), e_1^n \rangle$.

Problem 5. Prove that

$$D_n^{(m)}(z;q,q^{-1}) = q^{-m\binom{n}{2}} \frac{e_n[Z[mn+1]_q]}{[mn+1]_q}. \tag{113}$$

The right-hand side of (113) is equal to $\nabla^m e_n(z)|_{t=q^{-1}}$ by [5, Thm. 5.1]. The Catalan specialization

$$\langle D_n^{(m)}(z;q,q^{-1}), e_n \rangle = C_n^{(m)}(q,q^{-1}) \tag{114}$$

has been shown in [21, 22].

We remark that by the Cauchy formula,

$$\langle e_n[Z[mn+1]_q], h_\mu \rangle = q^{n(\mu')} \prod_i \begin{bmatrix} mn+1 \\ \mu_i \end{bmatrix}_q = \sum_{\substack{\lambda \subseteq ((mn+1)^n) \\ T \in \mathrm{SSYT}(\lambda+(1^n)/\lambda,\, \mu)}} q^{|\lambda|}. \tag{115}$$

Then (113) asserts that this last expression is equal to

$$[mn+1]_q \cdot \sum_{\substack{\lambda \subseteq m\delta_n \\ T \in \mathrm{SSYT}(\lambda+(1^n)/\lambda,\, \mu)}} q^{|\lambda| + \mathrm{dinv}_m(T)}. \tag{116}$$

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